UNITED STATES NAVAL POSTGRADUATE SCHOOL

ESTIMATING MEAN RELIABILITY GROWTH

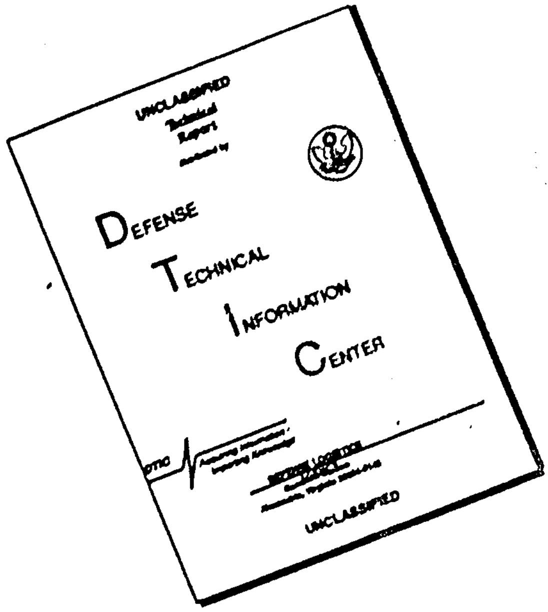
by

Peter W. Zehna

January 1966
Technical Report/Research Paper No. 60

Distribution of this document is unlimited

DISCLAIMER NOTICE



THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

UNITED STATES NAVAL POSTGRADUATE SCHOOL
Monterey, California

Rear Admiral E. J. O'Donnell, USN
Superintendent

Dr. R. F. Rinehart,
Academic Dean

ABSTRACT:

A model is defined wherein corrective action may be accounted for in improving the estimation of reliability over the usual nominal success ratio. Probabilities for correcting any one of K failure modes which may arise are assumed known within the structure of a multinomial sampling procedure. Mean reliability is defined as a function of the unknown probabilities attached to the failure modes, the problem being to estimate this mean. Other measures of current reliability are defined. Three different estimators of mean reliability are defined and analyzed from the point of view of unbiasedness. Explicit expressions for the bias are derived and compared numerically for a wide variety of choices for the unknown parameters. Several problem areas for further research are identified and partial formulations of some of these are discussed.

This task was supported by: Special Projects, Code SP-114

Prepared by: P. W. Zehna

Approved by:
J. R. Borsting
Chairman, Department of
Operations Analysis

Released by:
C. B. Menneken
Dean of
Research Administration

U. S. Naval Postgraduate School Technical Report, Research Paper No. 60
January 1966

UNCLASSIFIED

ESTIMATING MEAN RELIABILITY GROWTH

1. Introduction

A problem of considerable importance in current reliability studies is that of accounting for changes in reliability that result from various actions designed to modify a part or a system. Such modifications may range from design changes in the early states of development to corrective action taken to remove modes of failure that have been observed in a testing program at a later stage of development. Any model in which it is assumed that such modifications never decrease the reliability (probabability of successful operation) has come to be titled "reliability growth."

Despite the importance of the problem and the interest in solutions, very little published research on the topic of reliability growth exists. A brief list of some papers on the subject is given in the bibliography. In one reference [3] the writer, along with others, has developed one model to account for reliability growth and several estimations problems are discussed there. This same model is the main concern of the present report and is repeated here for the sake of completeness.

Suppose that an item or a system is to be tested and each test may result in success with probability p_o or exactly one of K fixed, but otherwise unspecified, modes of failure. The parameter p_o is referred to as the <u>initial reliability</u> and we denote the probability of failure of type i by q_i for $i=1, 2, \ldots, K$. Thus, $p_o + \sum_{i=1}^{K} q_i = 1$ and if we assume that N fixed, mutually independent tests are to be performed, the underlying probability model is that of the multinomial distribution with parameters $N, p_o, q_1, \ldots, q_K$. Accordingly, we denote the number of observed successes in N tests by N_o while N_i is used to denote the number of observed failures of type i. Thus, $N_o, N_1, \ldots, N_K$ are (marginally) binomial random variables subject to the restriction $\sum_{i=0}^{N} N_i = N$

By fixing K in the preceding formulation it is tacitly assumed that no new failure modes are ever introduced. Also, it is assumed that no corrective action is to be taken until all N tests are performed so that our procedures will be based on fixed sample size experiments. Having performed N tests, it is assumed that each observed failure mode can be classified as to type and that an attempt is then made to remove that mode of failure. However, it is not assumed that a failure mode is necessarily removed once it is observed. Indeed, subject to experience with the item or system, it is assumed that there is a known probability a_i of removing the i^{th} mode of failure given that corrective action is taken which in turn is always taken if and only if the failure mode is observed.

Under the above model, there are at least three measures of current reliability that are of interest each being appropriate at possibly different stages of development. First, and the most natural, is the actual current reliability, say R, which exists after the tests and after the corrective action takes place. Prior to testing, R is a random variable of course and may be written as follows. Let $y_i = 1$ if $N_i > 0$ and 0 otherwise, so that y_i is a random variable that accounts for whether or not the i^{th} failure mode occurs. Also, let $x_i = 1$ if corrective action on the i^{th} failure mode actually removes that mode of failure and 0 otherwise. We may then write,

$$(1.1)\quad R = p_0 + \sum_{i=1}^{K} x_i y_i q_i$$

In this form we see that current reliability is the initial reliability plus any failure probability that has been observed and actually removed. Reliability is thus not increased by a particular mode if either it is not observed or it is observed but not removed.

The quantity R which we have defined above is of primary interest after the complete testing program when the actual current reliability is desired. One perfectly straightforward way of estimating R is to perform N' additional tests observing N_o' successes and use the usual success ratio N_o'/N' as an estimate. It this is feasible, such a procedure certainly is on safe grounds statistically speaking. If, however, the cost or availability of items prohibits this direct approach, it is necessary to adopt a cruder measure of current reliability and use the results of the N tests to draw inference about the amount of reliability growth. One such measure would be the conditional mean of R, conditioned on the observed values of N_i, $i=0, 1, \ldots, K$. This conditional mean, denoted p_o^*, is derived in [3] and is given by

$$(1.2) \quad p_o^* = p_o + \sum_{i=1}^{K} y_i a_i q_i$$

where y_i is defined above. Such an average has an advantage in that averaging is taken with respect to whether or not corrective action is successful as a function only of the failure modes that are observed. In this sense, p_o^* is not the true current reliability, that is success probability, as it is erroneously referred to in [2] but is already an "average" reliability. The analysis of both R and p_o^* is the main concern of Report No. 2 of this study [5] and will not be discussed further in this report.

A third measure of current reliability is the unconditional mean of the current reliability (which is the same as the expected value of p_o^*) and is an "average" taken over all possible outcomes of the experiment. This measure of reliability is relevant before testing and before corrective action. Such a quantity would be suitable for assessing the potential gain in reliability to be derived from a corrective action program. Being a true parameter in the strict sense of the word, it lends itself quite well to standard statistical estimation tools. It is shown in [3] that this mean reliability, denoted μ', is given by the formula,

$$(1.3)\quad \mu = p_o + \sum_{i=1}^{K} a_i q_i - \sum_{i=1}^{K} a_i q_i (1-q_i)^N$$

References [2] and [3] address themselves mainly to the problem of finding estimators for μ under the conditions stated in the above model. Several estimators are defined in both references and various properties (of both a positive and a negative nature) are discussed. Three of these are of interest in the present study and, for the sake of definiteness, the notation of [3] (which does not always agree with that of [2] even for the same quantity) will be adopted.

The maximum likelihood estimator for μ is denoted p_3 and is defined quite simply by,

$$(1.4)\quad p_3 = \frac{N_o}{N} + \sum_{i=1}^{K} a_i \frac{N_i}{N} - \sum_{i=1}^{K} a_i \frac{N_i}{N} \left(1 - \frac{N_i}{N}\right)^N$$

An exact expression for the expected value of p_3 was not available in [3] although approximate expressions were derived. This created some limitations in comparing p_3 with other estimators on an equitable basis. One of the purposes of the present study is to resolve this problem of exact expressions for the moments. The results are presented in the sections to follow.

Another estimator, derived in [3] to meet the requirements of unbiasedness, is denoted p_5 and is given by the expression,

$$(1.5)\quad p_5 = \frac{N_o}{N} - \sum_{i=1}^{K} \sum_{j=1}^{N-1} (-1)^j \frac{(j+1)}{N-j} a_i \binom{N_i}{j+1}$$

The corresponding bias of p_5, defined by $b(p_5) = E(p_5) - \mu$ is given quite simply by

$$(1.6)\quad b(p_5) = (-1)^N \sum_{i=1}^{K} a_i q_i^{N+1}$$

It should be noted that p_5 is not unbiased and, indeed, it is shown in [3] that no unbiased estimator for μ exists. However, the bias for p_5 may be so small as to be negligible and this will be verified in Section 3 to follow.

For reasons peculiar to the Navy, a third estimator for μ has been adopted by Special Projects and is extensively discussed in [2]. This estimator is here (and in [3]) denoted by p_6 and is defined by,

$$(1.7)\quad p_6 = \frac{N_o}{N} + \sum_{i=1}^{K} Z_i \frac{N_i}{N} \text{ where } Z_i = \begin{cases} a_i & \text{if } N_i > 1 \\ 0 & \text{otherwise} \end{cases}$$

The moments of p_6 are easily computed and the bias is accordingly given by

$$(1.8)\quad b(p_6) = -\sum_{i=1}^{K} a_i q_i^2 (1-q_i)^{N-1}$$

It should be observed that the bias of p_6 is always negative so that p_6 is a "conservative" estimator. However, it may be that the amount of bias is serious enough to discredit conservatism in some cases. Several samples admitting a wide variety of choice for the various parameters in the above model, are delineated in [2]. However, a simulated version of the random variable p_o^* is used as a reference point rather than μ, no results regarding p_5 are presented and, for reasons mentioned previously, the moments of p_3 are omitted from discussions. An examination of the behavior of all of these elements for the same examples constitutes another portion of the present study. Results are summarized in the following sections.

2. Maximum Likelihood Estimator

One of the common features of the two estimators p_5 and p_6 defined in Section 1 is that no credit is given to failure modes that occur only once. This is easily seen by examining equations 1.5 and 1.7 where, if $N_i=0$ for any given i, that term involving N_i vanishes and therefore does not affect the value of the estimator. For small sample sizes, this is somewhat

undesirable since such a procedure appears to ignore some of the information in the sample. This may be the price one pays for attempting to avoid overestimating μ which was a requirement constantly kept in mind in deriving p_5 and p_6. It should be noted that the maximum likelihood estimator does not have this particular feature and every occurrence of a failure mode is allowed to increase the estimate of μ. To see how the bias is affected we first compute the expected value of p_3.

Since $\left(1-\frac{N_i}{N}\right)^N = \sum_{k=0}^{N} \binom{N}{k} (-1)^k \frac{N_i^k}{N^k}$, we may write

$$p_3 = \frac{N_o}{N} + \sum_{i=1}^{K} a_i \frac{N_i}{N} - \sum_{i=1}^{K} a_i \frac{N_i}{N} \sum_{k=0}^{N} \binom{N}{k} (-1)^k \frac{N_i^k}{N^k}$$

$$= \frac{N_o}{N} - \sum_{i=1}^{K} \sum_{k=1}^{N} (-1)^k a_i \binom{N}{k} \frac{N_i^{k+1}}{N^{k+1}} .$$

Thus, from linearity,

$$E(p_3) = p_o - \sum_{i=1}^{K} \sum_{k=1}^{N} (-1)^k a_i \binom{N}{k} E\left(\frac{N_i^{k+1}}{N^{k+1}}\right).$$

But for $i=1, 2, \ldots, K$, N_i is binomial with parameters q_i and N so that, for each $k=1, 2, \ldots, N$,

$$E\left(N_i^{k+1}\right) = \sum_{j=0}^{N} j^{k+1} \binom{N}{j} q_i^j (1-q_i)^{N-j} .$$

Substituting in the preceding expression,

$$E(p_3) = p_o - \sum_{i=1}^{K} \sum_{k=1}^{N} \sum_{j=0}^{N} a_i \frac{\binom{N}{j}\binom{N}{k}}{N^{k+1}} q_i^j (1-q_i)^{N-j} (-1)^k j^{k+1}$$

$$= p_o + \sum_{i=1}^{K} \sum_{j=0}^{N} a_i \binom{N}{j} q_i^{\,j} (1-q_i)^{N-j} \sum_{k=1}^{N} \binom{N}{k} \left(\frac{-j}{N}\right)^{k+1} .$$

But, $\sum_{k=1}^{N} \binom{N}{k} \left(\frac{-j}{N}\right)^{k+1} = \frac{j}{N} - \frac{j}{N}\left(1 - \frac{j}{N}\right)^{N}$ so that,

$$E(p_3) = p_o + \sum_{i=1}^{K} \sum_{j=1}^{N} a_i \binom{N}{j} q_i^{\,j} (1-q_i)^{N-j} \frac{j}{N} \left(1 - \left(1-\frac{j}{N}\right)^{N}\right)$$

Also, $\frac{j}{N}\binom{N}{j} = \binom{N-1}{j-1}$ for $j \neq 0$ so we may write,

$$E(p_3) = p_o + \sum_{i=1}^{K} \sum_{j=1}^{N} a_i \binom{N-1}{j-1} q_i^{\,j} (1-q_i)^{N-j} \left(1 - \left(\frac{N-j}{N}\right)^{N}\right)$$

$$= p_o + \sum_{i=1}^{K} a_i \sum_{j=1}^{N} \binom{N-1}{j-1} q_i^{\,j} (1-q_i)^{N-j} - \sum_{i=1}^{K} \sum_{j=1}^{N} a_i \binom{N-1}{j-1} q_i^{\,j} (1-q_i)^{N-j} \left(\frac{N-j}{N}\right)^{N}$$

After some minor simplications, we finally write

$$(2.1) \quad E(p_3) = p_o + \sum_{i=1}^{K} a_i q_i - \sum_{i=1}^{N} \sum_{j=0}^{N-2} a_i \binom{N-1}{j} q_i^{\,j} (1-q_i)^{N-1-j} \left(\frac{N-1-j}{N}\right)^{N}$$

The bias of $\hat{p}_3$ is then immediate and may be written as,

$$(2.2) \quad b(p_3) = \sum_{i=1}^{K} a_i q_i (1-q_i)^{N} \left[1 - \sum_{j=0}^{N-2} q_i^{\,j} \binom{N-1}{j} (1-q_i)^{-1-j} \left(\frac{N-1-j}{N}\right)^{N}\right]$$

Before examining the magnitude of the bias $b(p_3)$ in comparison with other estimators, it is possible to provide an alternative form of (2.1) using

Stirling numbers of the second kind. This form may be more appropriate for some purposes than the one given. As before, finding $E(p_3)$ reduces to being able to write the general m^{th} moment of the binomial distribution explicitly. The expression of the latter in terms of Stirling numbers does not seem to be widely recorded and is stated below as a theorem for the record. With regard to the proof, we follow the notation of [4] so that, for positive integers k and r, $k^{(r)} = k(k-1) \text{---} (k-r+1) = \frac{k!}{(k-r)!}$ if $k \geq r$ and is zero if $k < r$; $\mathcal{S}_r^k$ denotes the r^{th} Stirling number of the second kind of order k.

<u>Theorem</u> Let X be binomial with parameters n and p.

Then, $E(X^m) = \sum_{r=1}^{m} n^{(r)} \mathcal{S}_r^m p^r$ for every positive integer m.

Note: If $m > n$, $E(X^m) = \sum_{r=1}^{n} n^{(r)} \mathcal{S}_r^m p^r$ since $n^{(r)} = 0$ if $r > n$.

<u>Proof</u>: Let m be any positive integer. By definition,

$$E(X^m) = \sum_{k=1}^{n} \binom{N}{k} p^k q^{n-k} k^m .$$

But, $k^m = \sum_{r=1}^{m} \mathcal{S}_r^m k^{(r)}$ so that,

$$E(X^m) = \sum_{k=1}^{n} \sum_{r=1}^{m} \binom{n}{k} p^k q^{n-k} \mathcal{S}_r^m k^{(r)}$$

Since $k^{(r)} = 0$ if $k < r$ we may simplify further to,

$$E(X^m) = \sum_{r=1}^{m} \sum_{k=r}^{n} \frac{n!}{k\,(n-k)!} , \frac{k!}{(k-r)!} \mathcal{S}_r^m p^k q^{n-k}$$

$$= \sum_{r=1}^{m} n^{(r)} \mathcal{S}_r^m p^r \sum_{\alpha=0}^{n-r} \binom{n-r}{\alpha} p^\alpha q^{n-r-\alpha}$$

$$= \sum_{r=1}^{m} n^{(r)} \mathcal{S}_r^m \, p^r \qquad \text{q.e.d.}$$

To write $E(p_3)$ in terms of Stirling numbers we expand (1.4) to obtain,

$$p_3 = \frac{N_0}{N} + \sum_{i=1}^{K} a_i \frac{N_i}{N} - \sum_{i=1}^{K} a_i \frac{N_i (N-N_i)^N}{N^{N+1}} \text{ so that,}$$

$$(2.3) \quad E(p_3) = p_o + \sum_{i=1}^{K} a_i q_i - \sum_{i=1}^{K} \frac{a_i}{N^{N+1}} E\left[N_i (N-N_i)^N \right]$$

Now let $Z_i = N_i (N - N_i)^N$ and $Y_i = N - N_i$ so that

$Z_i = (N-Y_i) Y_i^N = N Y_i^N - Y_i^{N+1}$. Since N_i is binomial with parameters N and q_i, it follows that Y_i is binomial with parameters N and $1-q_i$. By the theorem,

$$E(Z_i) = N E(Y_i^N) - E(Y_i^{N+1}) = \sum_{i=1}^{N} N \, N^{(r)} \mathcal{S}_r$$

$$= \sum_{r=1}^{N} \left[N N^{(r)} \mathcal{S}_r^N (1-q_i)^r - N^{(r)} \mathcal{S}_r^{N+1} (1-q_i)^r \right]$$

$$= \sum_{r=1}^{N} N^{(r)} (1-q_i)^r \left(N \mathcal{S}_r^N - \mathcal{S}_r^{N+1} \right)$$

Since, $\mathcal{S}_r^{N+1} = \mathcal{S}_{r-1}^N + r \mathcal{S}_r$ (see [4] p. 17) we can simplify to Stirling numbers of the same order whence,

$$E(Z_i) = \sum_{r=1}^{N} N^{(r)} (1-q_i)^r \left[(N-r) \mathcal{S}_r^N - \mathcal{S}_{r-1}^N \right].$$

Substituting in (2.3) we finally obtain,

$$(2.4) \quad E(p_3) = p_o + \sum_{i=1}^{K} a_i q_i - \sum_{i=1}^{K} \sum_{r=1}^{N} \frac{a_i}{N^{N+1}} N^{(r)} (1-q_i)^r \left[(N-r) \mathcal{S}_r^N - \mathcal{S}_{r+1}^N \right]$$

Returning to the bias of p_3 given in (2.2) it is difficult, because of the complexity of the expression, to make general statements. Certainly p_3 may both overestimate as well as underestimate μ. A simple example shows this. Suppose N=2, K=1, and $a_1=1$. Now, if $q_1=.9$, then $\mu=.991$ while if $q_1 = .1$ then $\mu = .919$. In either case, $E(p_3)$ is given by .975 so that in the first case $(p_1 = .9)$, $b(p_3) = -.016$ and in the second case $b(p_3) = +.066$. More cases are treated in the next section.

3. Numerical Comparisons

To gain further insight into the results of the last section as well as to compare these results with those previously obtained in [2] and [3], it was decided to examine special cases numerically. For this purpose the examples documented in [2] were used. Such examples allegedly cover a wide variety of cases that are of practical significance. The tabulated results may be found in the appendix, Section 5. An example is defined by specifying the parameters K, p_o, q_1, q_2, ..., q_K. Nine such specifications are given. However, in each example, the parameters a_1, a_2, ..., a_k as well as N are further specified to provide fifteen cases in all. In reality, then, 135 examples are treated in the appendix. For each of these examples μ, the moments of each of the three estimators, p_3, p_5, p_6 are recorded as well as the bias in each case. In addition, the value of p_o^* determined by computer simulations in [2] is given for each example.

We previously remarked in Section 1 that p_o^*, the conditional mean of the true current reliability is, prior to the experiment, a random variable. Even after the experimental results are known, moreover, the value of p_o^* still cannot be determined because of the unknown parameters p_o, q_1, ..., q_K which enter explicitly in its formula. It is shown in [3] that the variance of p_o^* converges to zero as N becomes infinite. Hence, for large N, the values of p_o^* (whatever the experimental outcomes) and μ, its mean, should not be significantly different. The tables of Section 5 show that

these two quantities differ by very little even for moderate values of N--at least for the examples treated. What this means, of course, is that any estimator we choose for estimating μ, a parameter, can effectively be used also as a predictor for p_o^*, a random variable.

As for the maximum likelihood estimator, p_3, the tables reveal that the bias is positive in practically every case investigated. In some cases, Examples 3 through 8, the amount of positive bias is serious enough to make its use doubtful. Of course, we are speaking here only of unbiasedness as a criterion for choice. Recalling the original problem, one is attempting to take credit for corrective action in updating reliability over the initial state of nature p_o. Positive bias indicates a tendency to take more credit than is due and such optimism can be very misleading as to the potential worth of such a testing program. From this point of view, p_3 has little to offer the experimenter. The result is not too surprising since maximum likelihood estimators tend to be biased. Moreover, it is difficult to justify the maximum likelihood criterion, for which p_3 is the optimal choice, as one to adopt in the present circumstances.

As far as unbiasedness alone is concerned, it is here reiterated as in [3] that p_5 is "effectively" unbiased. Indeed, in every single example treated, the bias of p_5 is zero to three decimal place computation. Not one case arose where the result was different from zero. Such a situation is not surprising for large values of N as brought out in [3] but for small values of N the same result is somewhat surprising and helps support p_5 as an important contender for use as an estimator for μ.

It was anticipated that p_6 would underestimate μ since it was defined in such a way as to have this property. The amount of bias is somewhat serious in several of the examples (notably 1, 2, 4, and 8). As remarked previously, the price for conservatism may be too high. Certainly we wish to avoid overestimating; at the same time we should not want to be unduly severe so that we certainly wish to take credit for corrective action when such credit is due.

As a final remark we note the following interesting result in the examples. As the a_i's decrease, it is noted that for fixed N, the bias approaches zero (from either side). This means that as our ability to remove the cause of a detected failure decreases so too our tendency to overestimate (or underestimate, as the case may be) decreases.

4. Topics for Further Study

It is concluded that the matter of unbiasedness for the model treated in this report is settled. p_5 is preferable to the three estimators examined and certainly p_3 should be rejected on this basis. However, as previously remarked, unbiasedness is but one criterion. It is well known that a biased estimator is preferable over an unbiased one if the variance of the former is sufficiently smaller than that of the latter. This suggests adopting mean squared error as a criterion and comparing p_3, p_5 and p_6 on this basis.

In [3] the variance for p_6 has been derived and is given explicitly at least up to higher order terms. No such expression is yet available for p_5 although some (umpublished) computer simulations carried out in connection with [3] indicate that the variance of p_5 decreases rapidly with N. Clearly, the variance for p_3 can be written down along the lines of the first moment as derived in Section 2 although the algebra involved may be somewhat unwiedly. In any event, numerical values can certainly be obtained for such as the examples treated in this report. With such tools at hand, the three estimators could then be compared on a mean squared error basis.

Since μ is a well-defined function of the unknown parameters involves, another problem suitable for investigation is that of finding a Bayes estimator for μ. Some a priori assumptions about p_o, q_1, ..., q_K would of course have to be made and the results judged accordingly. Such an estimator should then be compated with the other candidates as to unbiasedness, mean squared error, etc.

As yet, little progress has been made within this model in the direction of confidence interval estimation. In no small part, this is due to the complete lack of distribution theory with regard to the estimators treated. It would be most desirable to study the problem of finding a lower one-sided confidence interval for μ. Even approximate results would be beneficial to the present state of the art.

Still another problem worthy of investigation is a re-examination of the model itself. In spite of its reasonableness, some aspects of the model are somewhat confining. Most notably, the matter of allowing failures to accumulate until all N tests are performed may be intolerable in some practical situations. It may be far more reasonable to stop testing as soon as a failure is observed, take the necessary corrective action, then proceed as before until the next failure occurs. Such a program of testing would thus involve several stages. In a given stage, the sample obtained would be a sample from a geometric distribution (having observed Bernoulli trials to first failure) but the parameter changes from one stage to the next if corrective action is successful. Again, various quantities related to the growth in reliability could be examined under this model.

A model similar to that just outlined is presented in a report [1] which appeared recently in the literature. A brief examination of this report reveals several shortcomings which will need to be overcome before the usefulness of the results can be assessed. In any case, the work presented there should be more closely examined if further study along the lines presented above is pursued.

5. Appendix

The tables to follow summarize the numerical results which are analyzed in Section 3. The tables are self-explanatory with all of the notation consistent with that previously adopted in this report. The examples were limited to those available in [2] in order to avoid computer simulations needed for evaluating p_o^*. Otherwise, any number of further examples may be defined as in the tables and the corresponding entries easily computed.

EXAMPLE 1

$K = 9 \quad p_o = .10$

$q_1 = q_2 = \ldots = q_9 = .10$

	$a_1 = 1.0$					$a_1 = 0.8$					$a_1 = 0.6$				
N	5	10	25	50	100	5	10	25	50	100	5	10	25	50	100
p_o^*	.434	.649	.928	.993	1.00	.370	.549	.748	.815	.820	.313	.431	.594	.635	.640
μ	.469	.686	.935	.995	1.00	.395	.569	.768	.816	.820	.321	.452	.601	.637	.640
$E(p_3)$	.786	.836	.937	.988	1.00	.649	.686	.769	.810	.820	.511	.542	.602	.633	.640
$b(p_3)$	.317	.150	.002	# .007	0	.254	.120	.001	# .006	0	.190	.090	.001	# .004	0
$E(p_5)$	.469	.686	.935	.995	1.00	.395	.569	.768	.816	.820	.321	.452	.601	.637	.640
$b(p_5)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$E(p_6)$	.406	.651	.928	.995	1.00	.348	.541	.763	.816	.820	.286	.431	.597	.637	.640
$b(p_6)$	# .063	# .035	# .007	0	0	# .047	# .028	# .005	0	0	# .035	# .021	# .004	0	0

- indicates negative value

TABLE 1

EXAMPLE 2

$K = 10 \quad p_o = .10$

$q_1 = q_2 = .20 \qquad q_3 = q_4 = q_5 = .10 \qquad q_6 = \ldots = q_{10} = .04$

	$a_1 = 1.0$					$a_1 = 0.8$					$a_1 = 0.6$				
N	5	10	25	50	100	5	10	25	50	100	5	10	25	50	100
p_o^*	.519	.718	.902	.974	.997	.451	.589	.747	.796	.816	.349	.468	.584	.622	.639
μ	.529	.719	.905	.972	.997	.443	.596	.744	.798	.817	.357	.472	.583	.623	.638
$E(p_3)$	.804	.856	.936	.976	.994	.663	.705	.769	.800	.815	.522	.553	.602	.625	.636
$b(p_3)$	.275	.136	.031	.004	# .003	.220	.109	.025	.002	# .002	.165	.081	.019	.002	# .002
$E(p_5)$	.529	.719	.905	.972	.997	.443	.596	.744	.798	.817	.357	.472	.583	.623	.638
$b(p_5)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$E(p_6)$	.469	.692	.899	.971	.996	.396	.573	.739	.797	.817	.322	.456	.579	.623	.638
$b(p_6)$	# .060	# .027	# .006	# .001	# .001	# .047	# .023	# .005	# .001	0	# .035	# .016	# .004	0	0

- indicates negative value

TABLE 2

EXAMPLE 3

$K = 100 \quad p_0 = .10$

$q_1 = q_2 = \ldots = q_{100} = .009$

	$a_1 = 1.0$					$a_1 = 0.8$					$a_1 = 0.6$				
N	5	10	25	50	100	5	10	25	50	100	5	10	25	50	100
p_0^*	.139	.176	.282	.426	.636	.132	.162	.245	.358	.533	.124	.146	.210	.297	.421
μ	.140	.178	.282	.427	.636	.132	.162	.246	.362	.528	.124	.147	.209	.296	.421
$E(p_3)$	.713	.703	.719	.753	.814	.590	.583	.595	.623	.671	.468	.462	.471	.492	.528
$b(p_3)$	.573	.525	.437	.326	.178	.458	.421	.349	.261	.143	.344	.315	.262	.196	.107
$E(p_5)$	.140	.178	.282	.427	.636	.132	.162	.246	.362	.528	.124	.147	.209	.296	.421
$b(p_5)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$E(p_6)$	.132	.170	.276	.422	.632	.126	.156	.240	.358	.526	.119	.142	.205	.293	.419
$b(p_6)$	# .008	# .008	# .006	# .005	# .004	# .006	# .006	# .006	# .004	# .002	# .005	# .005	# .004	# .003	# .002

- indicates negative value

TABLE 3

EXAMPLE 4

$K = 100 \quad p_o = .10$

$q_1 = q_2 = .20 \quad q_3 = q_4 = q_5 = .10 \quad q_6 = \ldots = q_{10} = .004 \quad q_{11} = \ldots = q_{100} = .002$

	$a_i = 1.0$					$a_i = 0.8$					$a_i = 0.6$				
N	5	10	25	50	100	5	10	25	50	100	5	10	25	50	100
p_o^*	.502	.650	.777	.819	.840	.426	.545	.653	.676	.691	.337	.437	.515	.531	.544
μ	.494	.657	.788	.819	.839	.415	.545	.650	.675	.691	.836	.434	.513	.532	.544
$E(p_3)$	.796	.841	.905	.928	.936	.657	.693	.744	.762	.769	.518	.545	.583	.597	.602
$b(p_3)$	.302	.184	.117	.109	.097	.242	.148	.094	.087	.078	.182	.111	.070	.065	.058
$E(p_5)$	.494	.657	.788	.819	.839	.415	.545	.650	.675	.691	.336	.434	.513	.532	.544
$b(p_5)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$E(p_6)$	.441	.634	.785	.819	.839	.373	.527	.648	.675	.691	.305	.420	.511	.531	.544
$b(p_6)$	# .053	# .023	# .003	0	0	# .042	# .018	# .002	0	0	# .031	# .014	# .002	# .001	0

- indicates negative value

TABLE 4

EXAMPLE 5

$K = 100 \quad p_o = .10$

$q_1 = q_2 q_3 = .10 \quad q_4 = .003 \quad q_5 = \ldots = q_{10} = .020 \quad q_{11} \ldots = q_{100} = .005$

	$a_1 = 1.0$					$a_1 = 0.8$					$a_1 = 0.6$				
N	5	10	25	50	100	5	10	25	50	100	5	10	25	50	100
p_o^*	.246	.351	.503	.599	.714	.213	.304	.418	.499	.586	.184	.245	.338	.397	.464
μ	.246	.339	.479	.575	.682	.216	.292	.403	.480	.566	.187	.244	.328	.385	.566
$E(p_3)$	.718	.728	.770	.805	.840	.595	.603	.636	.664	.692	.471	.477	.502	.523	.692
$b(p_3)$	.472	.389	.291	.230	.158	.379	.311	.233	.184	.126	.284	.233	.174	.138	.126
$E(p_5)$	.246	.339	.479	.575	.682	.216	.292	.403	.480	.566	.187	.244	.328	.385	.566
$b(p_5)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$E(p_6)$	.221	.324	.474	.572	.681	.197	.279	.399	.478	.564	.173	.234	.324	.383	.564
$b(p_6)$	# .025	# .015	# .005	# .003	# .001	# .019	# .013	# .004	# .002	# .002	# .014	# .010	# .004	# .002	# .002

- indicates negative value

TABLE 5

EXAMPLE 6

$K = 1000 \quad p_o = .10$

$q_1 = q_2 = \ldots = q_{1000} = .0009$

	$a_i = 1.0$					$a_i = 0.8$					$a_i = 0.6$				
N	5	10	25	50	100	5	10	25	50	100	5	10	25	50	100
p_o^*	.104	.108	.120	.140	.177	.103	.106	.116	.132	.162	.102	.105	.112	.124	.147
μ	.104	.108	.120	.140	.177	.103	.106	.116	.132	.162	.102	.105	.112	.124	.146
$E(p_3)$	.706	.688	.680	.681	.689	.585	.570	.564	.565	.571	.464	.453	.448	.449	.453
$b(p_3)$	.602	.580	.560	.541	.512	.482	.464	.448	.433	.409	.362	.348	.336	.325	.307
$E(p_5)$	.104	.108	.120	.140	.177	.103	.106	.116	.132	.162	.102	.105	.112	.124	.146
$b(p_5)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$E(p_6)$	.103	.107	.119	.139	.177	.103	.106	.115	.131	.161	.102	.104	.112	.123	.146
$b(p_6)$	# .001	# .001	# .001	# .001	0	0	0	# .001	# .001	# .001	0	# .001	0	# .001	0

- indicates negative value

TABLE 6

EXAMPLE 7

$K = 100 \quad p_o = .60$

$q_1 = q_2 \ldots = q_{100} = .004$

	$a_1 = 1.0$					$a_1 = 0.8$					$a_1 = 0.6$				
N	5	10	25	50	100	5	10	25	50	100	5	10	25	50	100
p_o^*	.608	.615	.639	.672	.733	.606	.613	.631	.660	.706	.605	.609	.623	.643	.680
μ	.608	.616	.638	.673	.732	.606	.613	.631	.658	.706	.605	.609	.623	.644	.679
$E(p_3)$	.871	.864	.865	.872	.886	.816	.811	.812	.817	.829	.762	.758	.759	.763	.772
$b(p_3)$	.263	.248	.227	.199	.154	.210	.198	.181	.159	.123	.157	.149	.136	.119	.093
$E(p_5)$	.608	.616	.638	.673	.732	.606	.613	.631	.658	.706	.605	.609	.623	.644	.679
$b(p_5)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$E(p_6)$	.606	.614	.637	.671	.731	.605	.611	.629	.657	.705	.604	.609	.622	.643	.679
$b(p_6)$	# .002	# .002	# .001	# .002	# .001	# .001	# .002	# .002	# .001	# .001	# .001	0	# .001	# .003	# 0

- indicates negative value

TABLE 7

EXAMPLE 8

$K = 81 \quad p_o = .60$

$q_1 = .10 \quad q_2 = \ldots = q_6 = .03 \quad q_7 = \ldots = q_{31} = .004 \quad q_{32} = \ldots = q_{81} = .001$

	$a_i = 1.0$					$a_i = 0.8$					$a_i = 0.6$				
N	5	10	25	50	100	5	10	25	50	100	5	10	25	50	100
p_o^*	.667	.710	.774	.836	.879	.655	.685	.741	.790	.824	.644	.663	.709	.742	.769
μ	.664	.709	.784	.837	.881	.651	.687	.747	.790	.825	.639	.665	.710	.742	.768
$E(p_3)$	.883	.887	.908	.928	.946	.826	.830	.846	.862	.877	.770	.772	.785	.797	.808
$b(p_3)$	.219	.178	.124	.091	.065	.175	.143	.099	.072	.052	.131	.107	.075	.055	.040
$E(p_5)$	.664	.709	.784	.837	.881	.651	.687	.747	.790	.825	.639	.665	.710	.742	.768
$b(p_5)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$E(p_6)$	.653	.701	.780	.836	.880	.643	.681	.744	.789	.824	.632	.661	.708	.742	.768
$b(p_6)$	# .011	# .008	# .004	# .001	# .001	# .008	# .006	# .003	# .001	# .001	# .007	# .004	# .002	0	0

- indicates negative value

TABLE 8

EXAMPLE 9

$K = 100 \quad p_o = .90$

$q_i = \ldots q_{100} = .001$

	$a_i = 1.0$					$a_i = 0.8$					$a_i = 0.6$				
N	5	10	25	50	100	5	10	25	50	100	5	10	25	50	100
p_o^*	.900	.901	.902	.905	.910	.900	.901	.902	.904	.907	.900	.901	.902	.903	.905
μ	.900	.901	.902	.905	.910	.900	.901	.902	.904	.908	.900	.901	.901	.903	.906
$E(p_3)$	.967	.965	.965	.965	.966	.954	.952	.952	.952	.953	.940	.939	.939	.939	.939
$b(p_3)$	.067	.064	.063	.060	.056	.054	.051	.050	.048	.045	.040	.038	.038	.036	.033
$E(p_5)$	.900	.901	.902	.905	.910	.900	.901	.902	.904	.908	.900	.901	.901	.903	.906
$b(p_5)$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$E(p_6)$	.900	.901	.902	.905	.090	.900	.901	.902	.904	.908	.900	.901	.901	.903	.906
$b(p_6)$	0	0	0	0	# .001	0	0	0	0	0	0	0	0	0	0

- indicates negative value

TABLE 9

6. Bibliography

[1] Barlow, R. E. and E. Scheuer, "Reliability Growth during a Development Testing Program," RAND Corporation, Santa Monica, California

[2] Corcoran, W. T. and H. Weingarten, "Estimation of Reliability after Corrective Action on Observed Failure Modes," Special Projects Office, Department of the Navy, Washington, D. C.

[3] Corcoran, W. T. Weingarten, H. and P. W. Zehna, "Estimating Reliability after Corrective Action," Management Science, Vol. 10, No. 4, July 1964.

[4] Miller, Kenneth S., The Calculus of Finite Differences and Difference Equations, Henry Holt and Company, New York, N.Y.

[5] "Conditional Distribution of True Reliability after Corrective Action," Report No. 2 Subproject 3900-3 this contract.

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* U. S. Naval Postgraduate School Monterey, California	2a. REPORT SECURITY CLASSIFICATION unclassified 2b. GROUP x

3. REPORT TITLE

Estimating Mean Reliability Growth

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Task Progress, June 1965 - December 1965

5. AUTHOR(S) *(Last name, first name, initial)*

Zehna, Peter W.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
15 January 1966	28	5

8a. CONTRACT OR GRANT NO. b. PROJECT NO. 46058 c. Task Assignment 88423 d.	9a. ORIGINATOR'S REPORT NUMBER(S) TR-60 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. AVAILABILITY/LIMITATION NOTICES

"Qualified requesters may obtain copies of this report from DDC."

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Special Projects Office SP-114

13. ABSTRACT

A model is defined wherein corrective action may be accounted for in improving the estimation of reliability over the usual nominal success ratio. Probabilities for correcting any one of K failure modes which may arise are assumed known within the structure of a multinomial sampling procedure. Mean reliability is defined as a function of the unknown probabilities attached to the failure modes, the problem being to estimate this mean. Other measures of current reliability are defined.

Three different estimators of mean reliability are defined and analyzed from the point of view of unbiasedness. Explicit expressions for the bias are derived and compared numerically for a wide variety of choices for the unknown parameters. Several problem areas for further research are identified and partial formulations of some of these are discussed.

DD FORM 1 JAN 64 1473

unclassified

Security Classification

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Probability						
Statistical Estimation						
Reliability						
Reliability Growth						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through
____________________ ."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through
____________________ ."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through
____________________ ."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.

unclassified

Security Classification